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STRUCTURAL STABILITY THEOREMS FOR INTEGRABLE DIFFERENTIAL FORMS ON 3-MANIFOLDS

CÉSAR CAMACHO

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§1. STATEMENT OF RESULTS

LET M BE a compact, connected, orientable C^∞ Riemannian manifold of dimension n and $\vartheta^r(M)$ the space of all integrable 1-forms of class C^r endowed with uniform C^r topology., i.e. $\omega \in \vartheta^r(M)$ if and only if ω is C^r and $\omega \wedge d\omega = 0$. A singularity of $\omega \in \vartheta^r(M)$ is a point in M where ω vanishes. Denote the union of all singularities of ω by Sing(ω). By Frobenius Theorem ω defines a regular foliation of codimension one on M-Sing(ω).

A topological equivalence between forms $\omega, \eta \in \vartheta^r(M)$ is a homeomorphism h of M sending leaves of ω/M-Sing(ω) onto leaves of η/M-Sing(η) and Sing(ω) onto Sing(η).

The form $\omega \in \vartheta^r(M)$ is called C^s-structurally stable, $s \leq r$, if there is a neighborhood $N(\omega)$ in $\vartheta^s(M)$ such that any $\eta \in N(\omega)$ is topologically equivalent to ω.

The main problem concerning the stability of integrable forms is to characterize the elements of $\vartheta^r(M)$ which are C^r-structurally stable.

The first one to deal with this problem from the local point of view was I. Kupka[6] and more recently several people, [3, 13]. Roughly there are two types of singularities, those $x \in M$ for which $\omega_x = 0$, $d\omega_x \neq 0$ and those for which ω_x and $d\omega_x$ vanish simultaneously.

Let $x^0 \in M$ be a singularity of ω such that $d\omega_{x^0} \neq 0$. The form $d\omega$ induces near x^0 a regular foliation $\mathcal{F}(\omega)$ defined by the vector fields X with $i_X(d\omega) = 0$. This foliation has codimension two, it is tangent to the leaves of ω and Sing(ω) near x° is a union of leaves of $\mathcal{F}(\omega)$. It follows that there is a system of coordinates $(x_1, \ldots x_n)$ around x° such that $\omega = a_1(x_1, x_2)\,dx_1 + a_2(x_1, x_2)\,dx_2$.

Definition ([6, 8]). Let $S \subset M$ be a submanifold such that $\omega_x = 0$ and $d\omega_x \neq 0$ for some $x \in S$. Then S is called *normally hyperbolic* if for some $x^0 \in S$ there is a system of coordinates $(x_1, \ldots, x_n)$ in a neighborhood $V_0 \ni x^0$ such that $\omega/V_0 = a_1(x_1, x_2)\,dx_1 + a_2(x_1, x_2)\,dx_2$ and the vector field $Y_\omega(x_1, x_2) = -a_2(\partial/\partial x_1) + a_1(\partial/\partial x_2)$ has a hyperbolic singularity at x^0. When the real parts of the eigenvalues of $DY_\omega(x^0)$ have the same sign S is called *normally attractor*. If they have different sign S is called *normally saddle*.

By the argument above this definition is independent of $x^0 \in S$, codimension of S is two and the foliation induced by ω is locally a product of a line field in the plane with a foliation of codimension two.

Consider now a singularity $x^0 \in M$ such that $\omega_{x^0} = 0$ and $d\omega_{x^0} = 0$. Unlike the singularities of vector fields the local C^s-structural stability of ω at x^0 depends strongly on s. So for instance there are C^1-stable germs of forms ω for which the 1-jet $J^1[\omega]_{x^0}$ is nondegenerate and also germs for which $J^1[\omega]_{x^0} \equiv 0$ is a C^2-stable property. These singularities are contained in the following definition.

Definition. Let ω be an integrable form on a 3-manifold and x^0 a singular point of ω such that $d\omega_{x^0} = 0$. Then x^0 is called hyperbolic if (i) $J^1[\omega]_{x^0} = dg$ and g is a nondegenerate quadratic function, or (ii) $J^1[\omega]_{x^0} \equiv 0$ and $J^2[\omega]_{x^0}$ can be written in some system of coordinates (x_1, x_2, x_3) where $x^0 = 0$ as:

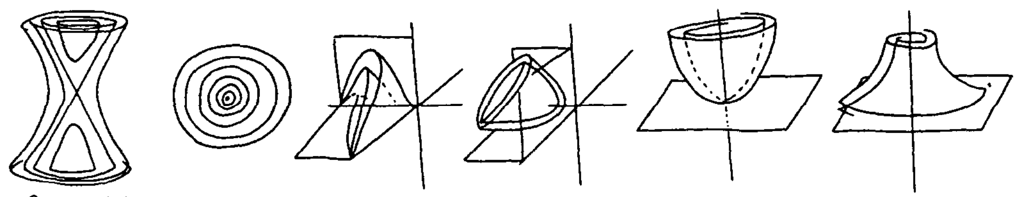
$$J^2[\omega]_{x^0} = \lambda_1 x_2 x_3\, dx_1 + \lambda_2 x_1 x_3\, dx_2 + \lambda_3 x_1 x_2\, dx_3 \qquad \lambda_i \neq \lambda_j \quad \text{for} \quad i \neq j,$$

or

$$J^2[\omega]_{x^0} = (\alpha x_1 + \beta x_2) x_3\, dx_1 + (-\beta x_1 + \alpha x_2) x_3\, dx_2 + \gamma (x_1^2 + x_2^2)\, dx_3$$

with $\alpha, \beta, \gamma \neq 0$.

Singularities of type (i) are very well known by the work of Reeb[11] even in the n-dimensional case (see also [8]). We will call x^0 a cone point if the index of g is one and center point otherwise. Singularities of type (ii) are C^2-stable but not C^1-stable and were studied in [7]. We will call them 2nd order real or complex singularities according to whether $J^2[\omega]_{x^0}$ has the first or second canonical form above. The leaf structure of $J^2[\omega]_{x^0}$ is the orbit structure of a hyperbolic linear action of the group $\mathbf{R}^2$ on $\mathbf{R}^3$[1]. There are three topological types given in Fig. 1.



Cone point   Center point   2nd order real singularities   2nd order complex singularities only one topological class

Fig. 1.

From now on M will denote a compact 3-manifold and ω an integrable form on M. Let L be a noncompact leaf of ω. A limit point of L is an accumulation point in M of a divergent sequence in L with the leaf topology. The union of all these points will be denoted $\lim(L)$. Write also $\lim(\omega) = \bigcup_L \lim(L)$ where L is any noncompact leaf of ω. In what follows write $\omega \in \vartheta^r_{i_1,\dots,i_k}(M)$ to indicate that $\omega \in \vartheta^r(M)$ satisfies properties $(i_1), \dots, (i_k)$. Define:

(1) All leaves of ω have abelian fundamental group and $\mathrm{Sing}(\omega)$ is a p-dimensional, $p \leq 1$, compact polyhedron whose vertices are hyperbolic singularities and 1-sides normally hyperbolic curves closed or not.

For $\omega \in \vartheta_1{}^r(M)$ write $\mathrm{Sing}(\omega) = \overline{\mathrm{sadd}(\omega)} \cup \overline{\mathrm{attr}(\omega)} \cup \mathrm{cent}(\omega)$. Where $\mathrm{sadd}(\omega) = \bigcup_{i \in I} s_i$ and s_i is a cone point or a normally saddle curve; $\mathrm{attr}(\omega) = \bigcup_{j \in J} a_j$ and $(a_j)_j \in J$ are the normally attractor curves of ω; $\mathrm{cent}(\omega)$ is the subset of center points.

If $\omega \in \vartheta_1{}^r(M)$ write $S(\omega)$ to denote the union of leaves L such that $\gamma \subset \lim(L) \subset \mathrm{Sing}(\omega)$ and $\gamma \subset \mathrm{sadd}(\omega)$. Also put $C(\omega)$ to denote the union of $\mathrm{Sing}(\omega)$ with all nonsimply connected leaves and $\mathscr{L}(\omega) = C(\omega) \cup S(\omega)$.

(2) $\mathscr{L}(\omega)$ is a compact polyhedron of dimension $p \leq 2$ without boundary and $\lim(\omega) \subset \mathscr{L}(\omega)$.

THEOREM A. *Let $\omega \in \vartheta^r_{12}(M)$, $r \geq 2$ and R a connected component of M-$\mathscr{L}(\omega)$. Then (R, ω) is one of the following types:*

(*i*) *Either $R \approx S^2 \times (0, 1)$ or $R \approx S^2 \times S^1$ and all leaves of ω / R are homeomorphic to S^2. Moreover no $\sigma \in \mathrm{Sing}(\omega) \cap \partial R$ is an attractor.*

(ii) *Either $R \approx D^2 \times (0, 1)$ or $R \approx D^2 \times S^1$* where $D^2 = \{(x_1, x_2) \in \mathbf{R}^2;\ x_1^2 + x_2^2 < 1\}$ *and all leaves of ω / R are homeomorphic to D^2. For any leaf L in R, $\lim(L) \subset \overline{\mathrm{attr}(\omega)}$. Moreover no $\sigma \in \mathrm{Sing}(\omega) \cap \partial R$, $\sigma \notin \lim(L)$ is an attractor.*

(iii) *$R \approx D^2 \times S^1$ and $\omega / \bar{R}$ is a (singular) Reeb component i.e. $\dim(\partial R) = 2$ and there is a vanishing cycle. This is a continuous family of paths α_t: $S^1 \to \bar{R}$, $0 \leq t \leq 1$,* *such that* $\alpha_0(S^1) \subset \partial R$ *and* $\alpha_t(S^1) \subset L_t$, $t > 0$, *where* L_t *is a leaf of* ω/R. *Moreover* α_t: $S^1 \to L_t$, $t > 0$, *is null homotopic and* α_0: $S^1 \to \partial R$ *is not.*

(iv) $R \approx D^2 \times S^1$, $\dim(\partial R) = 2$ *and* $\operatorname{attr}(\omega) \cap \partial R \neq \phi$.

To grant the stability of the above decomposition we impose further conditions on the behavior of ω.

A *saddle connection* is a nonsimply connected leaf L with $\lim(L) \subset \operatorname{Sing}(\omega)$ and such that any connected component of $\lim(L)$ contains elements of $\operatorname{sadd}(\omega)$. An *alternate connection* is a leaf L such that (a_1) $\lim(L) \subset \operatorname{Sing}(\omega)$ and contains a closed simple continuous curve $\gamma = \bigcup_{i=1}^{n} \overline{s_i}$ where each s_i is a normally hyperbolic curve, $\overline{s_i} \cap \overline{s_{i+1}} \neq \phi$ and $s_n = s_1$. (a_2) There are at least two s_i: s_{i_1}, s_{i_3} of saddle type and two attractors s_{i_2}, s_{i_4} in alternate order i.e. $i_1 < i_2 < i_3 < i_4$.

(3) There are no alternate connections and saddle connections have nontrivial linear holonomy.

Let π be a 1-form on an open subset $U \subset \mathbf{R}^2$. Suppose there are points $p_1, \ldots, p_n$; $p_n = p_1$, which are singularities of saddle type, i.e. there is a system of coordinates (x_1, x_2) near p_i, $x_1(p_i) = x_2(p_i) = 0$ such that $\pi = \mu_i x_2\, dx_1 + \lambda_i x_1\, dx_2 + R$, $\lim_{x \to 0} R(x)/|x| = 0$, λ_i, $\mu_i > 0$. Give an orientation to the integral curves of π and let $\mathscr{C}_0$ be a closed path formed by saddle separatrices joining p_i to p_{i+1} from $i = 1$ to $i = n-1$ along the positive sense. The possibility $p_{i+1} = p_i$ is not excluded.

Definition. The curve $\mathscr{C}_0$ is called a *simple cycle* if $\sum_{i=1}^{n-1} \sigma_i \neq 0$ where $\sigma_i = \lambda_i - \mu_i$.

Let $\overline{AB}$ be an open segment transversal to the integrals of π with end point A in a saddle separatrix. Let $C \in \overline{AB}$ such that for any $x \in \overline{AC}$ the positive integral through x intersects $\overline{AB}$ again. We will see in (4.3) that either the α or ω-limit set of $x \in \overline{AC}$ near A is $\mathscr{C}_0$ provided $\mathscr{C}_0$ is simple. This motivates the following hypothesis.

(4) Let $R \approx D^2 \times S^1$. Then any nonsimply connected leaf in ∂R has nontrivial linear holonomy. If any leaf in ∂R is simply connected there is an embedding e: $S^1 \times (-1, 1) \to M$, $U = e(S^1 \times (-1, 1))$, $\mathscr{C}_0 = e(S^1 \times 0) \subset \partial R$, in general position with ∂R such that $\mathscr{C}_0$ is a simple cycle of ω/U.

Our main result is:

THEOREM B. *Any* $\omega \in \vartheta^r_{1234}(M)$, $r \geq 2$, *is* C^2-*structurally stable.*

In §2 we give examples of these foliations. It is easy to find nonstable examples when one of the conditions (1)–(4) is not fulfilled.

The structural stability of a form ω on M is sometimes granted by the stability of ω in an open submanifold $M_1 \subset M$ even when ω is not stable in $M - M_1$. In §2.e. we show an example where this occurs.

All singularities we have been considering with the exception of centers and cones appear as lower dimensional orbits of actions of the group $\mathbf{R}^2$. The global behavior in this case is much more restricted than for foliations, thus the above theorems provide us with corollaries concerning the structural stability of $\mathbf{R}^2$-actions on 3-manifolds. These are given in §5.

I would like to thank Alcides Lins Neto and Airton de Medeiros for many conversations on the subject as well as the referee for valuable suggestions.

§2. GLOBAL EXAMPLES

(a) Let Y be a structurally stable vector field on a compact 2-manifold N with no periodic orbits and $f: N \to N$ an orientation preserving diffeomorphism sending orbits of Y to orbits of Y. Assume that for any singularity p of Y, p is a fixed point of f and $Df(p)$, $DY(p)$ admit a common invariant splitting of $T_p(N)$ with eigenvalues

$(\lambda_i)_{i=1}^2(\mu_i)_{i=1}^2$ such that $\mu_2 \log|\lambda_1| \neq \mu_1 \log|\lambda_2|$. Considering the product foliation on $S^2 \times [0,1]$ and identifying points by the relation $(x,0) \sim (f(x),1)$ obtain a foliation of $N \times S^1$ satisfying the hypothesis of Theorem B. An example of such a pair (Y,f) can be easily obtained on S^2 by taking two linear commuting automorphisms $\tilde{Y}$, $\tilde{f}$ on $\mathbf{R}^3$. With an appropriate choice of eigenvalues the projection $\pi: \mathbf{R}^3 - 0 \to S^2$, $\pi(x) = x/|x|$, will induce a pair (Y,f) satisfying the above inequality at any singular point.

(b) A Reeb foliation with singularities on S^3.

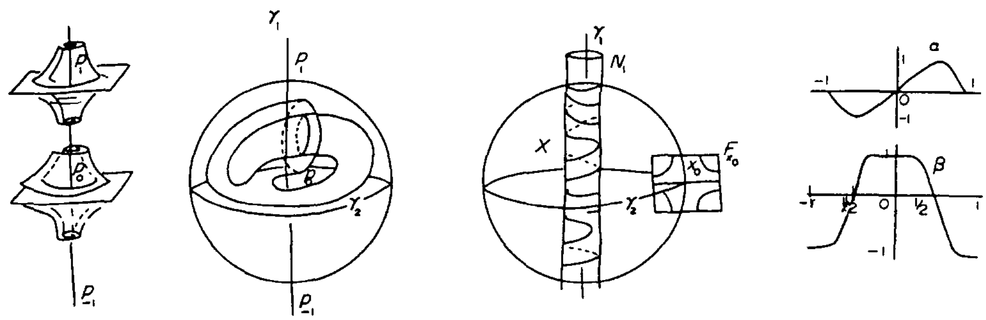


Fig. 2.

Let $B = \{x \in \mathbf{R}^3;\ x_1^2 + x_2^2 + x_3^2 \leq 1\}$ be the 3-ball, γ_1 the x_3-axis and N_1 a tubular neighborhood of γ_1. After a change of coordinates we can assume that $N_1 \cap \partial B = \{x \in N_1:\ x_3 = \pm 1\}$. Define ω on N_1 by $\omega = \alpha(x_3)(x_1 + x_2)\,dx_1 + \alpha(x_3)(-x_1 + x_2)\,dx_2 + \beta(x_3)(x_1^2 + x_2^2)\,dx_3$, where α and β are C^∞ functions as in Fig. 2. The singular points $p_i = (0,0,i)$, $i = -1, 0, 1$, are all complex hyperbolic of 2^{nd} order. The leaf intersection with ∂N_1 are the orbits of a vector field X with periodic orbits at $x_3 = 0$, ± 1. These orbits are hyperbolic and their Poincaré map is linear with eigenvalues $e^{2\pi}$, $e^{-2\pi}$ respectively. Let γ_2 be the closed curve $x_1^2 + x_2^2 = 1$, $x_3 = 0$ and N_2 a tubular neighborhood of γ_2. Again assume that $N_2 \cap \partial B = \{x \in N_2;\ x_1^2 + x_2^2 = 1\}$. The fibers of N_2 are intersections of the 2-planes containing the x_3-axis with N_2. For $p \in \gamma_2$ call F_p the fiber over p. Let $x_0 = (0,1,0) \in \gamma_2$. By a translation we can take $x_0 = (0,0) \in F_{x_0}$ and $A = -x_2(\partial/\partial x_2) + x_3(\partial/\partial x_3)$ a vector field on F_{x_0}. The linear diffeomorphism $f(x_2,x_3) = e^{2\pi}(x_2,x_3)$ commutes with A. Lift f to a fiber preserving flow f_t on N_2 such that $f_1/F_{x_0} = f$. The sets $\bigcup_t f_t(\theta)$ where θ is an orbit of A are leaves of an integrable form ω on N_2 with a hyperbolic singular curve γ_2. So far ω is defined on $N_1 \cup N_2$. To extend ω to a neighborhood V of $K = \partial B \cup \gamma_1 \cup \{x_3 = 0\}$ it only remains to define ω on $V - N_1 \cup N_2$ as a regular foliation. This is possible since the leaf intersection of ω with ∂N_2 yields a flow Y with periodic orbits θ_i, $\theta_0 = \{x_3 = 0\} \cap \partial N_2$, $\theta_1 \cup \underline{\theta}_1 = \partial N_2 \cap \partial B$ and Y near θ_i is differentiably equivalent to X near p_i. Let $V_1 \subset V$ be a neighborhood of K such that $B - V_1 = T_1 \cup T_2$ is the union of two solid tori with ∂T_i $i = 1, 2$ transverse to ω. Putting $T_1 \approx D^2 \times S^1$ the intersections of the leaves of ω with ∂T_1 is a foliation equivalent to the foliation by circles $\partial D^2 \times \{\theta\}$, $\theta \in S^1$. Thus we extend ω to T_1 as a foliation by 2-discs. Similarly for T_2. Glueing two copies of B along ∂B we obtain an example on S^3. This example does not embed in an $\mathbf{R}^2$-action with hyperbolic compact orbits.

(c) Linear actions on S^3. Given two linear commuting vector fields A_1 and A_2 on $\mathbf{R}^4$ we induce via the projection π: $R^4 - 0 \to S^3$, $\pi(x) = x/|x|$ two commuting vector fields X_1, X_2 on S^3. Let Ω be a volume on S^3. Under "hyperbolicity" conditions on the eigenvalues of A_1 and A_2, [1], the 1-form $\omega = i_{X_1 \wedge X_2}(\Omega)$ satisfies the hypothesis of Theorem B. The following pictures describe the leaf structure on B^3.

(d) Take A_1, A_2 as in (c) but with all eigenvalues complex. Under nondegeneracy conditions on the eigenvalues of A_1, A_2 the form ω has two normally attractor closed curves of singularities γ_1, γ_2. Depending on a relation of linear dependence over the

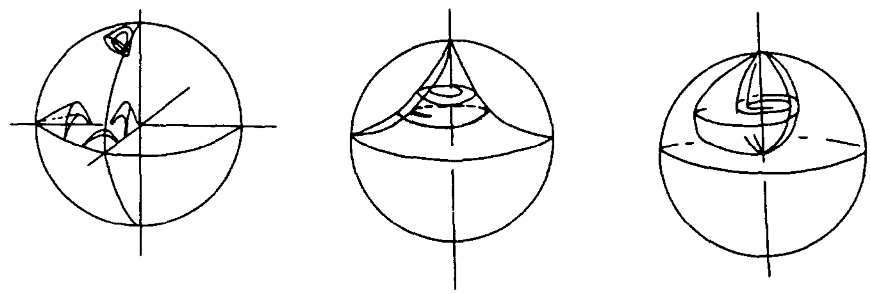


Fig. 3.

rationals of the eigenvalues of A_i, [1, 10], there are two possibilities: (1) $\lim(\omega) = \gamma_1 \cup \gamma_2$, no γ_i, $i = 1, 2$, is hyperbolic (§5) and all leaves are cylinders (2) $\lim(\omega) = S^3$ no γ_i is hyperbolic and all leaves are planes.

(e) Consider S^3 as union of two solid tori $T_i = D^2 \times S^1$, $i = 1, 2$ joined along the boundary by glueing parallels of ∂T_1 with meridians of ∂T_2. Let Y be a structurally stable C^2 vector field on D^2, transverse to ∂D^2, with no periodic orbits and only one singular point. Define on T_1 the product foliation $\mathcal{F}_1$ whose leaves are products of S^1 times the orbits of Y. This foliation is unstable[4]. In T_2 consider a regular foliation $\mathcal{F}_2$ transverse to ∂T_2 leaving as intersection the meridians of ∂T_2. All leaves of $\mathcal{F}_2$ cutting ∂T_2 accumulate on a compact leaf L bounding a regular Reeb component. Take L with nontrivial linear holonomy. Then $\mathcal{F}_2$ is stable. This implies that $\mathcal{F}_1 \cup \mathcal{F}_2$ is C^2-stable on S^3.

§3. PROOF OF THEOREM A

The following preliminary lemmas are concerned with the local structure of ω near a singularity.

(3.1) LEMMA. *Assume ω is a C^2-integrable form on an n-dimensional manifold and x_0 a singularity of ω. Suppose ω is written near x_0 as $\omega = dg + Q$ where g is a nondegenerate quadratic function at x_0 and $\lim_{x\to x_0}(Q(x)/|x|) = 0$. There is a neighborhood $U \ni x_0$ such that x_0 is an isolated singularity of ω/U and either* (1) *g has index or coindex zero and all leaves of ω/U are compact simply connected, or* (2) *g has index $\neq 0$, n and there is an integral cone of ω/U passing through x_0.*

Proof. The proof of (1) follows from a well known theorem of Reeb[11]. The proof of (2) is in [8].

Let $\mathrm{Sing}_2(\omega) = \mathrm{Sing}_2^{\mathbf{R}}(\omega) \cup \mathrm{Sing}_2^{\mathbf{C}}(\omega)$ be the set of 2nd order hyperbolic singularities of ω written as union of the subsets of real and complex type. Let $x_0 \in \mathrm{Sing}_2(\omega)$ and Ω_M a volume form on M. Define $X = \mathrm{rot}\,\omega$ by $d\omega = i_X(\Omega_M)$.

(3.2) LEMMA. *Let $\omega \in \vartheta^r(M)$, $r \geq 2$, $x_0 \in \mathrm{Sing}_2(\omega)$, and $T_{x_0}M = \bigoplus_{i=1}^{m} E_i$ the eigenspace decomposition of $DX(x_0)$. There is a neighborhood $U \ni x_0$ and C^{r-1}-submanifolds $S_i(x_0) = S_i(x_0; \omega)$ of U tangent to E_i at x_0, $\dim S_i(x_0) = \dim E_i$ such that* (i) *if $x_0 \in \mathrm{Sing}_2^{\mathbf{R}}(\omega)$, then $m = 3$ and $\mathrm{Sing}(\omega) \cap U = \bigcup_{i=1}^{3} S_i(x_0)$;* (ii) *if $x_0 \in \mathrm{Sing}_2^{\mathbf{C}}(\omega)$, then $m = 2$ and say $S_2(x_0) - \{x_0\}$ is a cylindrical leaf of ω with nontrivial linear holonomy and $\mathrm{Sing}(\omega) \cap U = S_1(x_0)$;* (iii) *the singular point x_0 is an isolated singularity of $d\omega$ in U and for any $S_i(x_0) \subset \mathrm{Sing}(\omega)$, $S_i(x_0) - \{x_0\}$ is normally hyperbolic.*

Proof. We can take $M \subset \mathbf{R}^3$ as a compact neighborhood of $x_0 = 0 \in \mathbf{R}^3$. Then $X = \mathrm{rot}\,\omega$ is the vector field $d\omega = i_X(dx_1 \wedge dx_3 \wedge dx_3)$. The invariant manifolds of X at x_0 are integral submanifolds of ω: In fact, let $W^s(x_0)$ be the local stable manifold of X

at x_0 defined via the flow X_t in a neighborhood $U \ni x_0$ as the set of points x for which $X_t(x) \to x_0$ as $t \to +\infty$. From the definition of X obtain $i_X(\omega) = 0$ and $\mathcal{L}_X(\omega) = i_X(d\omega) + d(i_X\omega) = 0$. Given $x \in W^s(x_0)$ and $v \in T_xW^s(x_0)$, $\omega_x(v) = \omega_{X_t(x)}$ $(DX_t(x) \cdot v)$. But as $t \to +\infty$ $X_t(x) \to 0$ and $DX_t(x) \cdot v \to 0$. Therefore $\omega_x(v) = 0$. Similarly for the unstable manifold $W^u(x_0)$.

Assume dim $W^u(x_0) = 1$. Let E_1 be the eigenspace of the positive eigenvalue of $DX(x_0)$ and $E_2' \oplus E_3'$ the invariant subspace of the eigenvalues of $DX(x_0)$ with negative real part. In coordinates $x = (x_1, x_2, x_3) \in E_1 \oplus E_2' \oplus E_3'$ we have $\omega = a_1(x)\, dx_1 + a_2(x)\, dx_2 + a_3(x)\, dx_3 = q + R$ with q in canonical form and

$$R = (R_1, R_2, R_3) \lim_{x\to 0} \frac{R_i(x)}{|x|^2} = 0, \qquad i = 1, 2, 3.$$

The singularities of ω in $W^s(x_0)$ are the intersections of the zeros of $a_1(x)$ with $W^s(x_0)$. If the eigenvalues of $DX(0)$ are real $a_1(x)$ has the form $a_1(x) = \lambda_1 x_2 x_3 + R_1(x)$. Then $\mathrm{Sing}(\omega) \cap W^s(x_0)$ is a union of two C^{r-1} curves $S_2(x_0)$, $S_3(x_0) \subset W^s(x_0)$ meeting transversally at x_0. Clearly $S_i(x_0)$ is tangent to E_i. If some eigenvalue of $DX(0)$ is nonreal then $a_1(x) = \gamma(x_2^2 + x_3^2) + R_1(x)$. So $\mathrm{Sing}(\omega) \cap W^s(x_0) = \{x_0\}$. Then take $S_2(x_0) = W^s(x_0)$. The verification that $S_2(x_0) - \{x_0\}$ has nontrivial linear holonomy is straightforward. Define $S_1(x_0) = W^u(x_0)$. Then $S_1(x_0) \subset \mathrm{Sing}(\omega)$: Let β_i the real part of the ith eigenvalue of $DX(0)$. We have $\beta_1 > 0 > \beta_2, \beta_3$ and $\beta_1 + \beta_2 + \beta_3 = 0$ as X is volume preserving. Let $\rho = \max\{|\beta_2|, |\beta_3|\}$ and $0 < \epsilon < (\beta_1 - \rho)/2$. Given $x \in W^u(x_0)$ and $v \in T_xM$, $\|\omega_x(v)\| = \|\omega_{X_t(x)}(DX_t(x)v)\| \leq \|\omega_{X_t(x)}\|\, \|DX_t(x)v\| \leq K.\ \exp[(\beta_1 - \epsilon)t] \cdot \exp[-(\rho + \epsilon)t]|v|$ for t large enough. Therefore $\omega_x(v) = 0$.

The singular points of $d\omega$ and X are the same. Thus x_0 is an isolated singularity of $d\omega$. Normal hyperbolicity of each $S_i(x_0) - \{x_0\} \subset \mathrm{Sing}(\omega)$ follows immediately from the expression of ω. From this obtain that outside $\bigcup_i S_i(x_0)$ there are no singularities of ω.

(3.3) LEMMA. *Let γ be a normally saddle singular curve. There is a neighborhood $U \supset \gamma$ and integral C^{r-1} submanifolds $S_i(\gamma) = S_i(\gamma; \omega)$, $i = 1, 2$, of dimension two of ω/U intersecting transversally along γ.*

Proof. Immediate from the definitions.

The following is a well known lemma.

(3.4) TRIVIAL HOLONOMY LEMMA. *Let $\mathcal{F}_0$ be a foliation of codimension one, F_0 a leaf of $\mathcal{F}_0$ and $K_0 \subset F_0$ a compact simply connected subset. There is a neighborhood $V \supset K_0$ and a diffeomorphism ξ_0: $V_0 \times (-1, 1) \to V$, $V_0 = V \cap F_0$, such that $\xi_0^*(\mathcal{F}_0)$ is the foliation induced by p_2: $V_0 \times (-1, 1) \to (-1, 1)$, $p_2(x, t) = t$, and $\xi_0(x, 0) = x$ for any $x \in V_0$.*

(3.5) *Proof of Theorem A.* Assertion (i) is an immediate consequence of Reeb stability theorem [11] when there is a compact leaf in R. Suppose that $\lim(L) \subset \overline{\mathrm{Sing}(\omega)}$ for some leaf L in R. Then $\lim(L) \subset \overline{\mathrm{attr}(\omega)}$. Let T be a neighborhood of $\overline{\mathrm{attr}(\omega)}$ such that ∂T is a differentiable surface transverse to the leaves of ω. Then by (2) $L \cap \partial T$ is a closed curve lying in a connected component τ of $\partial T - \mathcal{L}(\omega)$. By the trivial holonomy lemma all integrals of ω/τ are closed. So there is a curve l intersecting any leaf of ω/τ in one point. Clearly $R = \mathrm{sat}(\tau) = \mathrm{sat}(l)$ where $\mathrm{sat}(\tau)$ is the union of leaves touching τ. So R is homeomorphic to $D^2 \times I$ with $I = (0, 1)$ or S^1. Since all leaves in R have the same limit set any $\sigma \in \mathrm{Sing}(\omega) \cap \partial R - \lim(L)$ lies in $\mathrm{sadd}(\omega)$ or in $\mathrm{Sing}_2^{\mathbf{R}}(\omega)$.

Suppose now that $\lim(L)$ contains a leaf in ∂R. Then by well known arguments there is a closed curve $\bar\beta$ in R transverse to the leaves of ω/R. We prove that $\mathrm{sat}(\bar\beta) = R$. Let $x \in R \cap \overline{\mathrm{sat}(\beta)}$ and W a neighborhood of ∂R such that $\bar\beta \cap W = \phi$. By (2) and the trivial holonomy lemma there is a neighborhood $\mathcal{U} \ni x$ such that for any leaf L with $L \cap \mathcal{U} \neq \phi$, $L \cap \mathcal{U}$ is connected and $L - \mathcal{U} \subset W$. Thus $\bar\beta$ intersects all leaves in $\mathcal{U}$ and so $x \in \operatorname{sat}(\tilde{\beta})$. Since $\operatorname{sat}(\tilde{\beta})$ is clearly open, this shows that $\operatorname{sat}(\tilde{\beta}) = R$. By modifying $\tilde{\beta}$ if necessary we can suppose it cuts each leaf in R precisely once. Therefore $R \approx D^2 \times S^1$.

We proceed to show now that R is a Reeb component provided no $\sigma \in \operatorname{Sing}(\omega) \cap \partial R$ is an attractor. Let $\mathcal{N}$ be a vector field transverse to the leaves of ω, vanishing only at the singular points of ω. Near singular points $\mathcal{N}$ can assume a simple form (see (4.6)). Let R_ϵ be the set of points in R at a distance of ∂R bigger than ϵ along the integrals of $\mathcal{N}$. For $\epsilon > 0$ small enough we can define a diffeomorphism $f\colon R_\epsilon \to R$ preserving the orbits of $\mathcal{N}$ with $f/R_{2\epsilon} = \text{identity}$. Clearly $\partial R_\epsilon \approx S^1 \times S^1$ and since no $\sigma \in \operatorname{Sing}(\omega) \cap \partial R$ is an attractor $\omega^* = f^*\omega$ defines a C^r regular foliation $\mathcal{F}^*$ admitting a continuous extension to ∂R_ϵ. As $\lim(\omega^*) = \partial R_\epsilon$, there is a closed path $\beta\colon S^1 \to \partial R_\epsilon$, $\beta(1) = x_0$, inducing a nontrivial holonomy map $I_{x_0} \supset U_\beta \xrightarrow{f_\beta} I_{x_0}$, $U_\beta \ni x_0$, where I_{x_0} is a segment in R_ϵ, with end point x_0, transverse to $\mathcal{F}^*$. Let $\alpha\colon S^1 \to \partial R_\epsilon$, $\alpha(1) = x_0$, be such that $[\alpha] \neq 0$, $[\alpha]$, $[\beta]$ are independent elements of $\pi_1(\partial R_\epsilon)$ and $I_{x_0} \supset U_\alpha \xrightarrow{f_\alpha} I_{x_0}$ its holonomy map. Again, since $\lim(\omega^*) = \partial R_\epsilon$, there are m, $n \in Z$, $m \neq 0$, such that $f_\alpha{}^m = f_\beta{}^n$ in $U_\alpha \cap U_\beta$. So $\gamma^* = \alpha^m \circ \beta^{-n}$, $f_{\gamma^*} = \text{identity}$, defines a vanishing cycle of $\mathcal{F}^*$ and we can assume γ^* is simple. Call γ the closed path in ∂R defined as the projection of γ^* along the orbits of $\mathcal{N}$. We claim that γ defines a vanishing cycle for $\omega/\bar{R}$. Indeed it only remains to show that γ is not null homotopic. Let $\Sigma(\gamma)$ be a 2-dimensional cross section based on $\gamma(S^1)$ and L a leaf of ω/R. Then $L \cap \Sigma(\gamma) = \bigcup_n \gamma_n$ where γ_n are closed curves $\gamma_n \to \gamma$. Call D_n the disc bounded by γ_n. Then $\lim n \to \infty$ area $D_n = \infty$. On the other hand, if γ is null homotopic, it bounds a simply connected 2-complex A with finite area. By Reeb stability theorem adapted to this situation $\lim n \to \infty$ area D_n = area A which is a contradiction.

§4. PROOF OF THEOREM B

Let d be a Riemannian metric on M. Given a subset $S \subset M$ write $T_\delta(S) = \{x \in M;\ d(x, S) < \delta\}$ to denote the neighborhood of radius δ of S.

(4.1) LEMMA. *Suppose $\omega \in \vartheta_1{}^r(M)$, $r \geq 2$. Given $\epsilon > 0$ there is a neighborhood $N_1(\omega)$ in the C^r-topology and for $\eta \in N_1(\omega)$* a diffeomorphism $f\colon M \to M$ C^{r-1} *ϵ-close to the identity such that* $f(\operatorname{Sing}(\omega)) = \operatorname{Sing}(\eta)$ *and corresponding singularities have the same type.*

Proof. First we show that there is $0 < \epsilon_1 < \epsilon$ and $N(\omega)$ such that if $\eta \in N(\omega)$ then $\operatorname{Sing}(\eta) \subset T_{\epsilon_1}(\operatorname{Sing}(\omega))$. It is evident from the local theory that this is true for a neighborhood $T_{\epsilon_1}(K)$ of the subset K of $\operatorname{Sing}(\omega)$ formed by all hyperbolic singular points and normally hyperbolic closed curves. For $\epsilon_1 > 0$ small enough each connected component c of $T_{\epsilon_1}(\operatorname{Sing}(\omega)) - T_{\epsilon_1}(K)$ contains precisely one normally hyperbolic segment $\gamma_c(\omega)$ of ω. So if η is near ω in the C^r-topology, $\operatorname{Sing}(\eta) \cap c = \gamma_c(\eta)$ is a segment and $\gamma_c(\eta) \to \gamma_c(\omega)$ as $\eta \to \omega$. Outside $T_{\epsilon_1}(\operatorname{Sing}(\omega))$ ω has no singularities so the same is true for η close to ω. The construction of f near K follows using Lemmas (3.1) to (3.3). Its extension to a neighborhood of $\operatorname{Sing}(\omega)$ follows by standard methods.

Let K_ω, $K_\eta \subset M$ be polyhedrons formed of leaves and singularities of ω and η respectively. We say that K_ω, K_η are isomorphic if there is a homeomorphism between K_ω and K_ν sending $\operatorname{Sing}(\omega) \cap K_\omega$ to $\operatorname{Sing}(\eta) \cap K_\eta$.

(4.2) LEMMA. *Let $\omega \in \vartheta_{123}^r(M)$. Given $\epsilon > 0$ there is a neighborhood $N_2(\omega) \subset N_1(\omega)$ such that for any $\eta \in N_2(\omega)$, $S(\eta) \subset T_\epsilon(S(\omega))$ and $S(\eta)$ and $S(\omega)$ are isomorphic.*

Proof. Let $L = L(\omega)$ be a leaf in $S(\omega)$. For $\epsilon > 0$ small enough $L \cap T_\epsilon(\lim L) = \bigcup_{i=1}^m l_i(\omega)$ with $l_i(\omega)$ leaf of $\omega/T_\epsilon(\lim L)$ and $m = 1$ or 2 according to whether L is simply connected or not.

There is $\gamma_1(\omega) \subset \lim(L)$ which is either a cone point or a normally saddle curve. Suppose $\gamma_1(\omega) \subset \lim(l_1(\omega))$. By (4.1) for each $\eta \in N_1(\omega)$ there is a singular curve $\gamma_1(\eta) \subset T_\epsilon(\gamma_1(\omega))$ of the same type of $\gamma_1(\omega)$. By (3.1), (3.3) there is a leaf $l_1(\eta)$ of $\eta/T_\epsilon(\gamma_1(\omega))$ such that $\lim(l_1(\eta)) \supset \gamma_1(\eta)$ and $l_1(\eta) \to l_1(\omega)$ as $\eta \to \omega$. Define $L_1(\eta)$ as the leaf of η containing $l_1(\eta)$.

If L is simply connected $\lim(L)$ is connected and it is a cone point or a closed curve $\bigcup_{j=1}^{n+1} \overline{\gamma_j(\omega)}$ where by (3) the $\gamma_j(\omega)$ $j = 1, \ldots, n$ are singular curves normally saddle and either $\gamma_{n+1}(\omega) \subset \overline{\operatorname{attr}(\omega)}$ or $\gamma_{n+1}(\omega) = \phi$. By the local structure of the $\gamma_j(\eta)$ there are neighborhoods $\mathcal{U}_j \supset \gamma_j(\eta)$ such that $L_1(\eta) \cap S_1(\gamma_j;\ \eta)$ or $L_1(\eta) \cap S_2(\gamma_j,\ \eta)$ is nonempty in $\mathcal{U}_j$ $j = 1, \ldots, n$ (see (3.3)). This together with the trivial holonomy Lemma imply that $\lim L_1(\eta) = \bigcup_{j=1}^{n+1} \overline{\gamma_j(\eta)}$.

If L is not simply connected then $m = 2$. Assume first that $\lim(l_2(\omega)) \not\subset \overline{\operatorname{attr}(\omega)} - \operatorname{Sing}_2{}^{C}(\omega)$. Then there is $\gamma_2(\omega) \subset \lim\ (l_2(\omega))$ such that $\gamma_2(\omega) \subset \operatorname{sadd}(\omega)$ or $\gamma_2(\omega) \in \operatorname{Sing}_2{}^{C}(\omega)$. As before there is $\gamma_2(\eta) \subset T_\epsilon(\gamma_2(\omega))$ of the same type as $\gamma_2(\omega)$ and by (3.2), (3.3) a leaf $l_2(\eta)$ of $\eta/T_\epsilon(\gamma_2(\omega))$ such that $\lim\ (l_2(\eta)) \supset \gamma_2(\eta)$ and $l_2(\eta) \to l_2(\omega)$ as $\eta \to \omega$. Let $L_2(\eta)$ be the leaf of η containing $l_2(\eta)$. We proceed to show that $L_1(\eta) = L_2(\eta)$. First we notice that the leaf $L(\omega)$ has nontrivial linear holonomy. In fact, if $\lim l_2(\omega) \subset \operatorname{sadd}(\omega) \neq \phi$ this is true by (3) and if $\lim(l_2(\omega)) \cap \operatorname{Sing}^{C}(\omega) \neq \phi$ by (ii) of (3.2). Let $\alpha = \alpha(\omega) \subset L(\omega) - l_1(\omega) \cup l_2(\omega)$ be a closed simple path representing a generator of $\pi_1(L)$ and $\Sigma(\alpha)$ a 2-dimensional transverse section to the leaves of ω passing through α. The integrals of $\omega/\Sigma(\alpha)$ define a line field with a hyperbolic closed curve α. Let $K_1(\omega) = \overline{L - l_2(\omega)}$ and $K_2(\omega) = \overline{L - l_1(\omega)}$. Then $K_i(\omega) \cap \Sigma(\alpha) = \alpha$ for $i = 1, 2$. Let η be close enough to ω, say $\eta \in N_2(\omega) \subset N_1(\omega)$, such that $\eta/\Sigma(\alpha)$ has an isolated closed integral $\alpha(\eta)$ and there are compact neighborhoods $K_i(\eta) \subset L_i(\eta)$ of $l_i(\eta)$ with $\alpha(\eta) \to \alpha(\omega)$, $K_i(\eta) \to K_i(\omega)$ as $\eta \to \omega$. Then $K_i(\eta) \cap \Sigma(\alpha)$ is a closed curve. Consequently $K_i(\eta) \cap \Sigma(\alpha) = \alpha(\eta)$ for $i = 1, 2$ and so $L_1(\eta) = L_2(\eta)$.

On the other hand if $\lim(l_2(\omega)) \subset \overline{\operatorname{attr}(\omega)} - \operatorname{Sing}_2{}^{C}(\omega)$ then either $\lim(l_2(\omega))$ is a closed singular curve or a graph with vertices 2nd order real singularities and the proof follows easily.

Consider now the 1-form π in $\mathcal{U} \subset \mathbf{R}^2$ as in §1 and $p_1, p_2, \ldots, p_n = p_1$ the singularities of π, all of saddle type. Let X be a vector field on $\mathcal{U}$ such that $\pi(X) = 0$ and assume that modulo a translation one has near p_i, $X = (\lambda_i x_1 + R_1, -\mu_i x_2 + R_2)$ with $x_1(p_i) = x_2(p_i) = 0$ and $\lim_{x\to 0} R_j(x)/|x| = 0$, $j = 1, 2$. Let $\mathscr{C}_0$ be a cycle formed by saddle separatrices of X joining p_i to p_{i+1} along the positive sense. The possibility $p_{i+1} = p_i$ is not excluded.

In a neighborhood of p_i consider two small closed segments $L_i(L_i')$ transverse to X with end points $q_i(q_i')$ in the stable (unstable) manifold of p_i. Assume there is a segment $\overline{q_1c} = q_1 + (u, 0) \subset L_1$, $0 < u < \epsilon$, such that for any $x \in \overline{q_ic}$ the positive trajectory of X encounter all segments L_i, L_i', $i = 1, 2, \ldots, n-1$ after a positive time. Let $f\colon \overline{q_1c} \to L_1$ be the first return map. Clearly $f \in C^r$.

(4.3) Lemma. *Let $\mathscr{C}_0$ be a simple cycle. Then for any $x \in \overline{q_1c}$ sufficiently near to q_1 either the α-limit set or the ω-limit set of x is $\mathscr{C}_0$.*

Proof. Call X_t the flow induced by X and let T be the first positive number such that $X_T(u) \in L_1$. Then

$$f'(u) = c(T, u)\exp\left(\int_0^T \sigma(X_t(u))\,dt\right)$$

where $\sigma(X_t(u)) = (\partial X_1/\partial x_1)(X_t(u)) + (\partial X_1/\partial x_2)(X_t(u))$ and as $u \to 0$ $c(T,\ u) \to 1$ and $X_T(u) \to q_1$. Now

$$\int_0^T \sigma(X_t(u))\,dt = \sum_{i=1}^{n-1} \int_{t_i}^{t_i'} \sigma(X_t(u))\,dt + \sum_{i=1}^{n-1} \int_{t_i'}^{t_{i+1}} \sigma(X_t(u))\,dt$$

where $t_1 = 0$, $X_{t_i}(u) \in L_i$, $X_{t_i'}(u) \in L_i'$. Moreover when $u \to 0$. $\int_{t_i'}^{t_{i+1}} \sigma(X_t(u))\,dt$ goes to a finite limit. Assume $\sum_{i=1}^{n-1} \sigma_i < 0$. Then there is $\epsilon > 0$ such that $\sum_{i=1}^{n-1} \sigma_i < -\epsilon$. For u sufficiently near to zero one has $\sigma(X_t(u)) < \sigma_i + \epsilon/n$ for $t_i < t < t_i'$. So

$$\sum_{i=1}^{n-1} \int_{t_i}^{t_i'} \sigma(X_t(u))\,dt < \sum_{i=1}^{n-1} \int_{t_i}^{t_i'} (\sigma_i + \epsilon/n)\,dt = \sum_{i=1}^{n-1} (\sigma_i + \epsilon/n)(t_i' - t_i)$$
$$\leq \sum_{i=1}^{n-1} (\sigma_i + \epsilon/n)\tau < \left(\epsilon + \sum_{i=1}^{n-1} \sigma_i\right)\tau$$

where $\tau = \max(t_i' - t_i)$. Since $\tau \to \infty$ $\lim_{u\to 0} f'(u) = 0$. Therefore for any $x \in \overline{q_1 a}$ sufficiently near q_1 one has $\lim_{m\to\infty} f^m(x) = q_1$. If $\sum_{i=1}^{n-1} \sigma_i > 0$ one proceeds similarly.

(4.4) LEMMA. *Let $\omega \in \vartheta_{1234}'(M)$. Given $\epsilon > 0$ there is a neighborhood $N_3(\omega) \subset N_2(\omega)$ such that for any $\eta \in N_3(\omega)$, $\mathscr{L}(\eta) \subset T_\epsilon(\mathscr{L}(\omega))$, $\mathscr{L}(\eta)$ is isomorphic to $\mathscr{L}(\omega)$ and $\lim(\eta) \subset \mathscr{L}(\eta)$.*

Proof. Let $L = L(\omega)$ be a leaf in $\mathscr{L}(\omega) - S(\omega)$. Then L is either compact or $\lim(L) \subset \overline{\mathrm{attr}(\omega)}$. By (3.2) (ii) or by (ii) of Theorem A and (4) the leaf L has nontrivial linear holonomy. From this obtain easily that for η close enough to ω there is a leaf $L(\eta) \subset T_\epsilon(L)$ homeomorphic to L. This together with the stability of $\mathrm{Sing}(\omega)$ and $S(\omega)$ shows that for η near ω, say $\eta \in N_3(\omega)$, there is an η-integral polyhedron $l(\eta) \subset T_\epsilon(\mathscr{L}(\omega))$ homeomorphic to $\mathscr{L}(\omega)$. It remains to show that $l(\eta) = \mathscr{L}(\eta)$ i.e. any leaf $L(\eta)$ of η outside $l(\eta)$ is simply connected and $\lim(L(\eta)) \subset l(\eta)$.

To simplify, assume $l(\eta) = \mathscr{L}(\omega)$ and let R be a connected component of $M - \mathscr{L}(\omega)$. If the leaves of ω in R are compact so will be those of η by Reeb's theorem, [11, p. 132], and so there is nothing to prove.

On the other hand if there is a leaf F of ω/R such that $\lim(F) \subset \overline{\mathrm{attr}(\omega)}$ then $\tau = R \cap T_\epsilon(\lim F)$ is an annulus or a torus and by Theorem A all leaves of ω/τ are closed. When τ is a torus the same holds for η/τ by the trivial holonomy lemma. Let $\partial\tau \neq \phi$. Then the singularities of ω on $\bar{\tau}$, if any, are intersections of normally saddle singular curves with $\partial\tau$. So the connected components of $\partial\tau$ are closed curves contained in cylindrical leaves of ω or they are cycles of saddle connections of $\omega/\partial T_\epsilon(\lim F)$.

Let (β_n) be a sequence of integrals of ω/τ such that $\lim_{n\to\infty}\beta_n \subset \partial\tau$. Let $L_n(\omega)$ be the leaf of ω containing β_n. Since there are no alternate connections $L_\infty(\omega) = \lim_{n\to\infty} L_n(\omega)$ is a simply connected polyhedron contained in $\overline{S(\omega)}$. By the stability of $S(\omega)$ there is a polyhedron $L_\infty(\eta)$ isomorphic to $L_\infty(\omega)$. By Reeb stability theorem adapted to this situation for any leaf $L(\eta)$ near $L_\infty(\eta)$, $L(\eta) - T_{\epsilon/2}(\lim F)$ is homeomorphic to a compact 2-disc, thus leaving as intersection with τ a closed curve. Using again the trivial holonomy lemma obtain that any integral of η/τ is closed. So for any leaf $L(\eta)$ in R $\lim L(\eta) \subset \overline{\mathrm{attr}(\eta)}$.

Suppose $\lim(F) \not\subset \overline{\mathrm{attr}(\omega)}$ for some leaf F in R. Then by (3.5) there is a closed simple path γ in ∂R and a cross section $\Sigma(\gamma)$ in R over γ such that the integrals of $\omega/\Sigma(\gamma)$ are of two types: (i) all singularities, if any, of $\omega/\Sigma(\gamma)$ are of saddle type and all integrals of $\omega/\Sigma(\gamma)$ are closed, or (ii) there is a singular point $a_0 \in \gamma$ which is an attractor for $\omega/\Sigma(\gamma)$. Consider (i). Let $S \subset \Sigma(\gamma)$ the set of points $x \in R$ such that the integral I_x of $\eta/\Sigma(\gamma)$ through x is closed and homotopic to zero in the leaf $L_x(\eta)$. Then $S \neq \phi$ and open by the trivial holonomy lemma. S is closed because if $I_{x_n} \to I_x$, $I_{x_n} \subset S$,

I_x will be closed and if I_{x_n} is not homotopic to zero in its leaf then $L_x(\eta)$ would be compact by Novikov theorem, [9, 5]. Consequently the holonomy map of any path β in ∂R would have periodic points contradicting (4) (see (4.3)). Therefore for any leaf $L(\eta)$ in R, $\pi_1(L(\eta))=0$ and $\lim L(\eta) \subset \partial R$. Consider (ii). Given a leaf $L(\omega)$ in R, $L(\omega) \cap \Sigma(\gamma) = \bigcup_n \gamma_n(\omega)$ $\gamma_n(\omega) \cap \gamma_{n'}(\omega) = \phi$ for $n \neq n'$ and $a^0 \subset \lim \gamma_n$ for any n. Let $\alpha^0 \subset \mathrm{Sing}(\omega)$ be such that $\alpha^0 \cap \Sigma(\gamma) = a^0$. Then $\alpha^0 \in \mathrm{Sing}_2{}^C(\omega)$ or $\alpha^0 \subset \mathrm{attr}(\omega)$ and it is homeomorphic to S^1. Now, for any leaf $L(\eta)$, $L(\eta) \cap \Sigma(\gamma) = \bigcup_n \gamma_n(\eta)$ and by (4), $\lim_{n\to\infty} \gamma_n(\eta) \subset \gamma$.

We prove now that any leaf $L(\eta)$ in R is simply connected. Let $L(\omega)$ be a leaf of ω. There is $\delta > 0$ and a neighborhood V of $L(\omega) - T_\delta(\partial R)$ where ω is trivial. Consequently for η near ω there is a neighborhood V', $L - T_\delta(\partial R) \subset V' \subset V$ trivial for η. So if μ is a closed path in a leaf $L(\eta)$ intersecting V', μ can be homotoped to a path contained in $T_\delta(\alpha^0)$. However $L(\eta) \cap \partial T_\delta(\alpha^0)$ is a simply connected curve. Thus μ is null homotopic in $L(\eta)$.

(4.5) Corollary. *The subset* $\vartheta^r_{1234}(M)$ *is open in* $\vartheta^r(M)$ *for* $r \geq 2$.

In what follows con(ω) denotes the set of cone points of ω.

(4.6) Lemma. *Let* $\omega \in \vartheta^r_{12}(M)$, $r \geq 2$. *There is a vector field* $\mathcal{N}$ *on* M *and a foliation* $\mathcal{F}$ *of a neighborhood* U *of* $\mathrm{Sing}(\omega) - \mathrm{con}(\omega) \cup \mathrm{cent}(\omega)$, *such that*: (1) $\mathrm{Sing}(\mathcal{N}) = \mathrm{Sing}(\omega)$, $\mathcal{N}$ *is transverse to the leaves of* ω *in* M-$\mathrm{Sing}(\omega)$, *tangent to* $\mathcal{F}$ *in* U *and* cent(ω), con(ω) *are hyperbolic singularities of* $\mathcal{N}$. (2) $\mathrm{Sing}(\mathcal{F}) = \mathrm{Sing}(\omega)$. *At a singular point* $\mathcal{F}$ *is given by the level surfaces of* $g \circ f$ *where* f *is* C^{r-1} *diffeomorphism and* g *a nondegenerate quadratic function. At a regular point* $\mathcal{F}$ *comes from a submersion over* $\mathrm{Sing}(\omega)$. (3) *For any connected component* R *of* $M - \mathcal{L}(\omega)$ *either* R *is a Reeb component and* $\pm\mathcal{N}$ *has a unique periodic orbit in* R *which is a hyperbolic attractor or any integral curve of* $\mathcal{N}/\bar{R}$ is a compact segment with extremities on ∂R, *intersecting all leaves of* ω/R.

Proof. Near cone and center points $\mathcal{N}$ is easily defined. Let $x_0 \in \mathrm{Sing}_2(\omega)$. In a local chart we can write $x_0 = 0 \in \mathbf{R}^3$, $\omega = q + R$ with $\lim_{x\to 0} R(x)/|x|^2 = 0$ and $q = \sum_{i=1}^{3} q_i$ in one of the forms:

(i) $q = \lambda_1 x_2 x_3\, dx_1 + \lambda_2 x_1 x_3\, dx_2 + \lambda_3 x_1 x_2\, dx_3, \qquad \lambda_i \neq \lambda_j, \quad i \neq j;$

or

(ii) $q = (\alpha x_1 + \beta x_2) x_3\, dx_1 + (-\beta x_1 + \alpha x_2) x_3\, dx_2 + \gamma(x_1{}^2 + x_2{}^2)\, dx_3, \quad \alpha, \beta, \gamma \neq 0.$

It is easy to see that the vector field $\mathrm{grad}\, q = \sum_{i=1}^{3} q_i(\partial/\partial x_i)$ is tangent to the level surfaces of the function $g(x_1, x_2, x_3) = (\lambda_2 - \lambda_3)x_1{}^2 + (\lambda_3 - \lambda_1)x_2{}^2(\lambda_1 - \lambda_2)x_3{}^2$ in case (i) or $g(x_1, x_2, x_3) = -\gamma(x_1{}^2 + x_2{}^2) + \alpha x_3{}^2$ in case (ii). There is by (3.2) a neighborhood $T_\epsilon(x_0) \ni x_0$ and a C^{r-1} diffeomorphism f: $T_\epsilon(x_0) \to \mathbf{R}^3$, C^{r-1} close to the identity such that $\mathrm{Sing}(f^*q) = \mathrm{Sing}(\omega)$. Define $\mathcal{N}_1 = df^{-1} \circ \mathrm{grad}\ q \circ f$. An easy computation shows that $\mathcal{N}_1$ is transverse to the leaves of ω. The foliation $\mathcal{F}$ is defined by the level surfaces of the functions $g \circ f$ on $T_\epsilon(\mathrm{Sing}_2(\omega))$ extended to $U - T_\epsilon(\mathrm{Sing}_2(\omega))$ by a submersion π_1: $U - T_\epsilon(\mathrm{Sing}_2(\omega) \to \mathrm{Sing}(\omega)$. In each component of $U - T_\epsilon(\mathrm{Sing}_2(\omega))$ there is an orthogonal system of coordinates $(x_1, x_2, x_3) \in D^2 \times S^1$ such that $\pi_1(x_1, x_2, x_3) = x_3$. Let $Y = Y_1(\partial/\partial x_1) + Y_2(\partial/\partial x_2)$ be defined by $i_Y(dx_1 \wedge dx_2) = \omega - \omega(\partial/\partial x_3)\, dx_3$. Then the vector field $\mathcal{N}_2 = -Y_2(\partial/\partial x_1) + Y_1(\partial/\partial x_2)$ is tangent to $\mathcal{F}$ and $\omega(\mathcal{N}_2) = |\mathcal{N}_2|^2$. Let $0 \leq \beta \leq 1$ be a C^∞ bump function in U equal to one in $T_{\epsilon/2}$ $(\mathrm{Sing}_2(\omega))$ and zero in $U - T_\epsilon(\mathrm{Sing}_2(\omega))$. Then $\mathcal{N} = \beta\mathcal{N}_1 + (1-\beta)\mathcal{N}_2$ satisfies (1) and (2).

As there are finite number of leaves in $\mathcal{L}(\omega)$, $\mathcal{N}$ extends to a neighborhood of $\mathcal{L}(\omega)$.

Let R be a connected component of $M - \mathscr{L}(\omega)$. If $R \approx D^2 \times (0, 1)$ then $\mathcal{N}$ extends using the product structure of R.

Let R be a Reeb component and e: $D^2 \times S^1 \to M$ the embedding whose image is R. By (3.5) there is an embedded 2-dimensional torus transverse to the leaves of ω and arbitrarily close to ∂R. Let $T_\delta = e(\partial D^2(1-\delta) \times S^1)$ be such a torus. Then $\mathcal{N}$ is transverse to T_δ and say, entering R. By (3.5) $\omega/e(D^2(1-\delta)\times S^1)$ is equivalent to the foliation p_2: $D^2(1-\delta)\times S^1 \to S^1$, $p_2(x, y) = y$. The extension of $\mathcal{N}$ with a hyperbolic attractor follows immediately.

On the other hand, if there is no transverse torus in R there must exist an element α of attr(ω) in ∂R. Let $\partial T_\epsilon(\alpha)$ be transverse to the leaves of ω and assume $\mathcal{N}$ is tangent to $\partial T_\epsilon(\alpha)$. Then $\Gamma = R \cap \partial T_\epsilon(\alpha) \approx S^1 \times (0, 1)$ where $\partial\Gamma = \Gamma_1 \cup \Gamma_2$ and Γ_i is a point of a closed curve. Moreover the integrals of ω/Γ spiral from Γ_1 to Γ_2. It is not hard to see that there is a submersion p: $R \to \Gamma$ leaving invariant the foliation ω. $\mathcal{N}$ is then extended to R such that on $R - T_\epsilon(\partial R)$, $\mathrm{d}p(\mathcal{N}) = \mathcal{N} \circ p$. Since all integrals of $\mathcal{N}/\Gamma$ are compact segments so are those of $\mathcal{N}/R$.

(4.7) *Proof of Theorem B.* Let $N_3(\omega)$ be the neighborhood given by (4.4). Fix $\eta \in N(\omega)$ and assume $\mathrm{Sing}(\eta) = \mathrm{Sing}(\omega)$, $C(\eta) = C(\omega)$, $S(\eta) = S(\omega)$. Let R be an elementary region given by (3.5). Proceed to define an equivalence h between $\omega/\bar{R}$ and $\eta/\bar{R}$ putting $h/\partial R$ = identity. For any $y \in R$ let $\mathcal{N}_y$ (resp. $L_y(\eta)$) be the integral of $\mathcal{N}$ (resp. η) passing through y.

If $R \approx D^2 \times (0, 1)$, fix $y_0 \in R$ and for any $y \in R$, $p(y) = L_y(\omega) \cap \mathcal{N}_{y_0}$. Then $h(y) = L_{p(y)}(\eta) \cap \mathcal{N}_y$ is the required homeomorphism.

If $R \approx D^2 \times S^1$ and $y \in R$ the leaf $L_y(\eta)$ intersects $\mathcal{N}_y$ in a sequence $(y_n)_{n\in \mathbf{Z}}$, $y_0 = y$, ordered by $\mathcal{N}$. Define f_η: $R \to R$ putting $f_\eta(y_n) = y_{n+1}$. Clearly f_η extends to the identity on ∂R. We proceed to find a fundamental domain for f_η. Let $\gamma \subset \partial R$ be a simple path and $\Sigma(\gamma) \subset R$ a 2-dimensional surface invariant by $\mathcal{N}$ with γ in its boundary. Let (γ_n) be the sequence of intersections of $L_y(\eta)$ with $\Sigma(\gamma)$ and $S_n \subset \Sigma(\gamma)$ the closed strip between γ_n and γ_{n+1}. Then for n large enough the part of $L_y(\eta)$ between γ_n and γ_{n+1} joined with S_n is a continuous surface transverse to $\mathcal{N}$ outside S_n. We approximate this surface by another S differentiable and everywhere transverse to $\mathcal{N}$. Let B be one of the connected components of $\bar{R} - \bar{S}$ intersecting ∂R. Then $D(\eta) = B - f_\eta(B)$ is a fundamental domain for f_η.

Let Θ be a simple closed curve in R intersecting once any leaf of η/R. If R is a Reeb component Θ is the periodic orbit of $\mathcal{N}$, (4.6). The identity map of Θ induces a homeomorphism h_0: $\mathscr{L}\epsilon(\omega) \to \mathscr{L}\epsilon(\eta)$ between leaf spaces. Let π_η: $\bar{R} \to \mathscr{L}\epsilon(\eta)$ be the projection. Then h_0 induces a homeomorphism h_1: $D(\omega) \to D(\eta)$ by $h_1(y) = \mathcal{N}_y \cap \pi_\eta^{-1}(h_0\pi_\omega L_y(\omega)) \cap D(\eta)$. Finally, for any $y \in R$—Θ there is a unique $n \in \mathbf{Z}$ such that $f_\omega{}^n(y) \in D(\omega)$. Then $h(y) = f_\eta{}^{-\eta} \circ h_1 \circ f_\omega{}^n(y)$ is the required homeomorphism.

§5. STRUCTURAL STABILITY OF $\mathbf{R}^2$-ACTIONS

Consider a C^1 action φ: $\mathbf{R}^2 \to \mathrm{Diff}^1(M)$ of the group $\mathbf{R}^2$ on M. This means that $\varphi(r_1 + r_2) = \varphi_1(r_1) \circ \varphi(r_2)$ and $\varphi(0)$ = identity. Write $\varphi_r = \varphi(r)$ and fix an increasing sequence (K_n), $\bigcup_{n\geq 1} K_n = \mathbf{R}^2$ of compact neighborhoods of $0 \in R^2$. A point $x \in M$ is called nonwandering if for any neighborhood V of x and $n_0 \in Z_+$ there is $r \notin K_{n_0}$ such that $\varphi_r(V) \cap V \neq \phi$. Call $\Omega(\varphi)$ the set of nonwandering points of φ.

A singular orbit of φ is one whose isotropy group is nonzero. It follows that any singular orbit is in $\Omega(\varphi)$.

The notion of hyperbolicity for compact orbits of actions of the group $\mathbf{R}^2$ was introduced in [1] as follows. Let $\gamma = \theta_x$ be a compact orbit of φ. Then the map ρ: $G \to \mathrm{Aut}(T_\gamma M)$ given by $\rho(x, v) = (\varphi_g(x), \mathrm{d}\varphi_g(x)v)$ defines a linear bundle action on $T_\gamma M$.

Definition. Suppose x is a fixed point of φ and $G = \mathbf{R}^2$ or $\mathbf{R} \times \mathbf{Z}$. Then x is called hyperbolic if there is an invariant splitting $T_xM = \bigoplus_{i=1}^{m} E_i$ such that ρ restricted to each component of $E_i - 0$ is transitive and for any $v_i \in E_i - 0$ the induced action $\rho/G_i(\rho)$: $G_i(\rho) \to \mathrm{Aut}(T_xM)$ is normally hyperbolic at E_i. Here $G_i(\rho)$ stands for the isotropy group of v_i by the action ρ.

Definition. Let $\gamma = \theta_x$ be a compact orbit of φ. Then θ_x is called hyperbolic if ρ leaves invariant a continuous splitting $T_\gamma M = T\gamma \bigoplus_{i=1}^{n} E_i$. Moreover if $x \in \gamma$, the induced linear action ρ: $G_x(\rho) \to \mathrm{Aut}\,(\bigoplus_{i=1}^{n} E_i)$ is hyperbolic.

The action φ is generated by two commuting vector fields X_1, X_2 i.e. $[X_1, X_2] = 0$. The orbits of φ are the integral submanifolds of the form $\omega = i_{X_1 \wedge X_2(\Omega M)}$ where Ω_M is a volume form on M.

The local stability of these actions was proved in [1].

THEOREM C. *Let φ: $\mathbf{R}^2 \to \mathrm{Diff}^1(M)$ be a C^1 action on a compact 3-manifold. Assume that*

(i) *$\Omega(\varphi)$ is an embedded compact polyhedron consisting of finitely many singular orbits.*

(ii) *Any compact orbit of φ is hyperbolic.*

(iii) *There are no alternate connections.*

Then φ is C^1-structurally stable.

Proof. Given a cylindrical orbit θ of φ, there is a linear combination $Z = aX + bY$ such that all orbits of Z in θ are periodic of period t_0. This means that under the hypothesis, $\partial\theta$ contains no singularities other than hyperbolic closed curves or 2$^{\text{nd}}$ order singularities of complex type. This implies by (ii) that θ has nontrivial linear holonomy. Therefore since any 2-orbit of θ in $\Omega(\varphi)$ is a cylinder the foliation defined by φ satisfies (3) and (4) of the main theorem. Moreover (i) implies (2). Finally the local stability of φ and the arguments of §4. finish the proof.

Using the Poincaré–Bendixson theorem for $\mathbf{R}^2$-actions given in [2] one obtains the following.

COROLLARY. *Let φ:$\mathbf{R}^2 \to \mathrm{Diff}^1(M)$ be a C^1-action on a compact simply connected 3-manifold M such that*

(i) *Any one-dimensional orbit of φ is embedded.*

(ii) *Any compact orbit is hyperbolic.*

(iii) *There are no alternate connections.*

Then φ is C^1-structurally stable.

Starting with actions χ_1: $\mathbf{R}^2 \to \mathrm{Aut}(\mathbf{R}^n)$, χ_2: $\mathrm{Aut}(\mathbf{R}^n) \to \mathrm{Diff}(M)$ one obtains $\mathbf{R}^2$-actions $\rho = \chi_2 \circ \chi_1$ on any manifold M where the group $\mathrm{Aut}(\mathbf{R}^n)$ acts. These so called linearly induced actions were introduced in [1] to define $\mathbf{R}^2$-actions with hyperbolic compact orbits on n-spheres.

A characterization of the C^1-structural stability of these actions on S^3 was given in [10]: a linearly induced $\mathbf{R}^2$-action on S^3 is structurally stable if and only if any compact orbit is hyperbolic. For these actions it is immediate that (i) and (iii) of the corollary always hold.

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Instituto de Mathemática Pura e Aplicada, Rio de Janeiro, Brasil